Group Representation Theory
Exercises

1 Representations

1. Let $G = C_4 \times C_2 = \langle \sigma, \tau \mid \sigma^4 = \tau^2 = e,\ \sigma\tau = \tau\sigma \rangle$. Consider the matrices

$$S = \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix} \qquad \text{and} \qquad T = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$

Verify that sending $\sigma \mapsto S$ and $\tau \mapsto T$ defines a representation of G. Now let

$$Q = \begin{pmatrix} i & 0 \\ 1 & 1 \end{pmatrix} \qquad \text{and} \qquad R = \begin{pmatrix} -1 & 0 \\ i+1 & 1 \end{pmatrix}$$

Verify that sending $\sigma \mapsto Q$ and $\tau \mapsto R$ also defines a representation of G. Show that S is conjugate to Q. Show that R is conjugate to T. Are these two representations equivalent?

2. Using the natural bases, write down:

   (a) The 3-dimensional permutation representation of S_3.

   (b) The regular representation of C_5.

   (It's enough to give the values on generators for each group.)

3. Recall that S_3 and D_3 are the same group. Draw an equilateral triangle in the plane with vertices at

$$v_1 = (1, 0), \quad v_2 = (-\frac{1}{2}, \frac{\sqrt{3}}{2}), \quad v_3 = (-\frac{1}{2}, -\frac{\sqrt{3}}{2})$$

From this picture we can construct a 2-dimensional representation of S_3.

   (a) In the standard basis, what matrix represents the permutation (123)?

   (b) In the standard basis, what matrix represents the permutation (23)?

   (c) Find a new basis in which the first of these matrices becomes diagonal. Write down the second matrix in this new basis.

4. Write down the regular representation of C_2 in the natural basis. Write down an equivalent matrix representation of C_2 in which all the matrices are diagonal.

5. Let $f : H \to G$ be a group homomorphism, and let

$$\rho : G \to GL(V)$$

be a representation.

   (a) Suppose ρ is a trivial representation. Show that $\rho \circ f$ is also trivial.
   (b) Suppose f is surjective. Show that if $\rho \circ f$ is a trivial representation then ρ must also be trivial.
   (c) Give an example of an f and a ρ such that $\rho \circ f$ is trivial but ρ is not trivial.

6. (a) Show that any 1-dimensional representation of a group G must be constant over conjugacy classes.
   (b) Recall that the group S_n is generated by transpositions, and that all transpositions are conjugate. Prove that S_n has exactly two 1-dimensional irreps.

7. *Advanced question:* Let G be a group, and $H \subset G$ be a subgroup of index k. Explain how we can use H to construct a representation of G of dimension k. If $G = S_n$ and $H = A_n$, what representation do we get?

8. (a) Prove Claim 1.4.2 from the notes. Prove that the composition of two G-linear maps is G-linear.
   (b) Prove Claim 1.4.9. from the notes.
   (c) Prove Claim 1.5.1 from the notes.

9. Let V be a vector space with basis $\{b_1, ..., b_n\}$. Let G be a subgroup of S_n, and let $\rho : G \to GL(V)$ be the permutation representation. Consider the vector

$$x = b_1 + b_2 + ... + b_n$$

   (a) Show that $\langle x \rangle \subset V$ is a subrepresentation. What 1-dimensional representation is it isomorphic to?

      So permutation representations are never irreducible! Find examples in the notes of specific permutation representations where we found this 1-dimensional subrepresentation.
   (b) Find a G-linear projection from V to $\langle x \rangle$. *Hint: look in the notes.*

10. Let $\rho_V : G \to GL(V)$ and $\rho_W : G \to GL(W)$ be two isomorphic representations of G. Prove that V is irreducible iff W is irreducible.

11. Let $G = D_4 = \langle \sigma, \tau \,|\, \sigma^4 = \tau^2 = e,\, \tau\sigma\tau = \sigma^{-1} \rangle$. There is a 3-dimensional representation of G

$$\rho : G \to GL_3(\mathbb{C})$$

defined by

$$\rho(\sigma) = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & -1 \\ -1 & 1 & 0 \end{pmatrix}, \qquad \rho(\tau) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

Find a 1-dimensional irrep U_1 of ρ. Can you find another one? Deduce that ρ can be decomposed as a direct sum

$$\mathbb{C}^3 = U_1 \oplus U_2$$

where U_2 is a 2-dimensional irrep. *NB: you don't need to find U_2.*

12. Let $G = \langle \mu \rangle$ be the *infinite* cyclic group, also known as $(\mathbb{Z}, +)$. Then

$$\rho(\mu) = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

defines a matrix representation of G. Find a 1-dimensional subrepresentation U of ρ, and show that there is no complementary subrepresentation to U. Where in the proof of Maschke's Theorem did we have to use the fact that G was finite?

13. *Advanced question:*

    (a) Suppose we want to think about representations of finite groups over a field $\mathbb{F}$ different from $\mathbb{C}$. What assumption do we need on $\mathbb{F}$ to make the proof of Maschke's Theorem work?

    (b) Let $\mathbb{F} = \mathbb{F}_2$. Let $G = C_2 = \langle \mu \mid \mu^2 = e \rangle$. Show that

    $$\rho(\mu) = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

    defines a representation of G. Find a 1-dimensional subrepresentation with no complementary subrepresentation.

14. a) Let $\rho_V : G \to GL(V)$ and $\rho_W : G \to GL(W)$ be two irreps of G, with $\dim V \neq \dim W$. Show that the only G-linear map from V to W is the zero map.

    b) Let $G = C_6$. How many irreps of G are there? How many of these irreps are faithful?

15. Prove Claim 1.6.5 from the notes. Using the same argument, prove Claim 1.6.7.

16. Let $g, h \in G$ be elements such that $gh = hg$. Let $\rho : G \to GL(V)$ be a representation. Prove that there exists a basis of V in which both $\rho(g)$ and $\rho(h)$ become diagonal matrices.

17. Let U, V and W be vector spaces, and let

$$\phi \in \mathrm{Hom}(U, V) \qquad \text{and} \qquad \psi \in \mathrm{Hom}(V, W)$$

Show that both maps

$$\circ\phi : \mathrm{Hom}(V, W) \to \mathrm{Hom}(U, W)$$
$$f \mapsto f \circ \phi$$

and

$$\psi\circ : \mathrm{Hom}(U, V) \to \mathrm{Hom}(U, W)$$
$$f \mapsto \psi \circ f$$

are linear. Deduce Claim 1.7.2.

18. Prove Claim 1.7.4.

19. Let $G = S_3$, with generators $\sigma = (123)$ and $\tau = (12)$. Let $\rho : G \to GL(V)$ be the 2-dimensional representation

$$\rho(\sigma) = \begin{pmatrix} \omega & 0 \\ 0 & \omega^{-1} \end{pmatrix} \qquad \rho(\tau) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

where $\omega = e^{\frac{2\pi i}{3}}$.

   (a) Write down the representation $\mathrm{Hom}(V, V)$ in the usual basis.

   (b) How do we know that $\mathrm{Hom}(V, V)^G$ must be 1-dimensional? Find a vector that spans it.

   (c) Find the decomposition of $\mathrm{Hom}(V, V)$ into irreps.

20. Let $G = D_5 = \langle \sigma, \tau \mid \sigma^5 = \tau^2 = e,\ \tau\sigma\tau = \sigma^{-1} \rangle$.

   a) Show that G has exactly two 1-dimensional representations.

   b) Find how many irreps of G there are (up to isomorphism), and find their dimensions.

21. Let $G = D_k = \langle \sigma, \tau \mid \sigma^k = \tau^2 = e,\ \tau\sigma\tau = \sigma^{-1} \rangle$.

   (a) Show that the number of 1-dimensional representations of G is 2 if k is odd, and 4 if k is even.

   (b) Find the dimensions of all irreps of G, for the cases $k = 6, 7$ and 8.

22. (a) Let G be any group, and let $V = \mathbb{C}^2$ be the two-dimensional trivial representation of G. Find a pair U, W of irreducible subrepresentations of V such that

$$V = U \oplus W$$

   Now find another subrepresentation W', different from W, such that we also have

$$V = U \oplus W'$$

   So although the irreps occuring in the two decompositions are isomorphic, they don't have to be the same subrepresentations.

   (b) *Advanced question:* Now let G be any group, and let U and W be any two non-isomorphic irreps of G. Let

$$V = U \oplus W$$

   Show that if W' is a subrepresention of V, and W' is isomorphic to W, then W' and W must be the same subrepresentation. *Hint: Show that any two G-linear injections from W to V must have the same image.*

23. Prove Claim 1.9.8. *Hint: Think about block diagonal matrices.*

24. Let $G = C_k$, so the irreps of G are $\rho_0, \ldots, \rho_{k-1}$. What is the dual of the irrep ρ_q? Which irrep do we get if we tensor ρ_q and ρ_r together?

25. Let V and W be representations of G. Pick bases for V and W, and find an isomorphism of representations between $V \otimes W$ and $W \otimes V$. Without picking bases, find an isomorphism of representations between $\mathrm{Hom}(V^*, W)$ and $\mathrm{Hom}(W^*, V)$.

26. Let $G = D_k = \langle \sigma, \tau \mid \sigma^k = \tau^2 = e,\ \tau\sigma\tau = \sigma^{-1} \rangle$. Let $V = \mathbb{C}^2$, and let $\rho_V : G \to GL(V)$ be the representation

$$\rho_V(\sigma) = \begin{pmatrix} \alpha & 0 \\ 0 & \alpha^{-1} \end{pmatrix}, \qquad \rho_V(\tau) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

where α is a kth root of unity. Let $W = \mathbb{C}$, and let ρ_W be the representation

$$\rho_W(\sigma) = 1, \qquad \rho_W(\tau) = -1$$

a) Verify that ρ_V is a representation.

b) Using the standard bases, write down the dual representation V^*, the representation $V \otimes W$, and the representation $\mathrm{Hom}(V, W)$.

2 Characters

27. Let $\rho : G \to GL_1(\mathbb{C})$ be a 1-dimensional representation, and let χ_ρ be its character. Show that $\chi_\rho(g)$ is a root of unity, for all $g \in G$.

28. Look back at Question 1, where we defined two representations of $C_4 \times C_2$. Show, by considering their characters, that the two representations are not equivalent.

29. Let $G = S_3$.

(a) Write down the three irreducible characters of G. *You only need to write down their values on each conjugacy class.*

(b) Let V be the 2-dimensional irrep of G. Find the characters of V^*, $V \otimes V$ and $\mathrm{Hom}(V, V)$.

(c) Write the character of $\mathrm{Hom}(V, V)$ as a linear combination of the irreducible characters. Check that your answer is consistent with your answer to Question 19.

30. Let X be a set with n elements, and let G be a subgroup of the group of all permutations of X. Let ρ be the associated n-dimensional permutation representation of G.

(a) Show that $\chi_\rho(g)$ equals the size of the set

$$\{x \in X,\ g(x) = x\}$$

(b) Show that the function

$$\xi : G \to \mathbb{C}$$

$$\xi(g) = \chi_\rho(g) - 1$$

is also a character of G. *Hint: Look at Question 9.*

31. Let $G = D_4 = \langle \sigma, \tau \mid \sigma^4 = \tau^2 = e,\ \tau\sigma\tau = \sigma^{-1} \rangle$.

  (a) Show that σ is conjugate to σ^3, that τ is conjugate to $\sigma^2\tau$, and that $\sigma\tau$ is conjugate to $\sigma^3\tau$. Deduce that G contains at most 5 conjugacy classes.

  (b) Write down the four irreducible characters of G corresponding to the four 1-dimensional irreps (see Question 21).

  (c) Show that G contains exactly 5 conjugacy classes.

32. Let $G = C_3 = \big\langle \mu \mid \mu^3 = e \big\rangle$.

  (a) Write down the three irreducible characters χ_0, χ_1, χ_2 of G corresponding to the three irreps U_0, U_1, U_2.

  (b) There's a 2-dimensional representation ρ of C_3 defined by

$$\rho(\mu) = \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix}, \qquad \rho(\mu^2) = \begin{pmatrix} -1 & -1 \\ 1 & 0 \end{pmatrix}$$

  Write down the character χ of ρ. Compute the inner product $\langle \chi | \chi_1 \rangle$. How many copies of U_1 occur in the irrep decomposition of ρ?

33. Let G be the alternating group $A_4 \subset S_4$. There are 4 conjugacy classes in A_4, they have representatives $(1), (123), (132), (12)(34)$, and sizes 1,4, 4, 3 respectively.

  (a) Find the number of irreps of G and their dimensions.

  (b) Let $\chi_4 : G \to \mathbb{C}$ be the function which is constant on each cycle type, and takes the values

$$\chi_4((1)) = 3 \qquad \chi_4((123)) = \chi_4((132)) = 0 \qquad \chi_4((12)(34)) = -1$$

  Using Question 30, show that χ_4 is a character of G. Prove that χ_4 is irreducible.

  (c) Based on what we've found so far, write down as much of the character table of G as you can. Using row and/or column orthogonality, write down equations for the remaining entries.

  (d) Find the values of all the irreducible characters on the class of $(12)(34)$.

  (e) Show that all remaining entries must be cube roots of unity (*hint: use Question 27*) and hence find the complete character table.

  (f) *Advanced question:* Why are your results consistent with the fact that there is a surjective homomorphism from A_4 to C_3?

34. Let $G = A_4$ again, and let $\chi_1, ..., \chi_4$ be the irreducible characters we found in Question 33. let $U_1, ..., U_4$ be the corresponding irreps of G. Find the decomposition of $U_4 \otimes U_4$ into irreps.

35. Prove the converse to Proposition 1.6.2, i.e. that if all the irreps of G are 1-dimensional then G must be abelian. *Hint: find the number of conjugacy classes in G.*

36. Use the column orthogonality relations to deduce Corollary 1.8.12. Now use them to show that in Proposition 2.1.8, we can deduce part (ii) from part (i).

37.

	g_1	g_2	g_3
$\lvert [g_i] \rvert$	1	2	3
$\lvert C_{g_i} \rvert$	6	3	2
χ_1	1	1	1
χ_2	1	1	-1
χ_3	2	x	y

Find the unknown values x, y in the above character table, (a) using row orthogonality, then (b) using column orthogonality.

3 Algebras and modules

38. Let $G = S_3 = \langle \sigma, \tau \,|\, \sigma^3 = \tau^2 = e,\, \tau\sigma\tau = \sigma^{-1} \rangle$. Let $A = \mathbb{C}[G]$ be the group algebra of G.

    a) What is the dimension of A? What is the unit element in A?

    b) Let a be the element

    $$a = e + \sigma + \tau \in A$$

    Compute a^2.

39. Show that $\mathbb{C}[G]$ is commutative iff G is abelian.

40. Find an isomorphism of algebras between $\mathbb{C}[C_3]$ and $\mathbb{C} \oplus \mathbb{C} \oplus \mathbb{C}$.

41. Let A and B be algebras. Show that the projection map $\pi : A \oplus B \to A$ is a homomorphism, but that the inclusion map $\iota : A \to A \oplus B$ is not.

42. Let A be any algebra. Show that the zero-dimensional vector space $\{0\}$ is an A-module.

43. *(Advanced question)* Define an A-module to be a vector space M together with a bilinear map

    $$\mu : A \times M \to M$$

    such that

    $$\mu(1_A, m) = m, \qquad \forall m \in M$$

    and

    $$\mu(ab, m) = \mu(a, \mu(b, m)), \qquad \forall a, b \in A \text{ and } m \in M.$$

    Prove that this definition is equivalent to the one given in the notes.

44. *(Advanced question)* Find an algebra A which is not isomorphic to its opposite algebra A^{op}.

45. State how many simple A-modules (up to isomorphism) there are, in the cases:

(a) $A = \mathbb{C}[S_3]$

(b) $A = \mathbb{C}[C_2] \oplus \mathbb{C}[S_3]$.

46. Show (directly) that the matrix algebra $\mathrm{Mat}_{2\times 2}(\mathbb{C})$ has no 1-dimensional modules. *Hint: Look at the relations between the standard basis elements.*

47. Let $A = \mathbb{C}[C_3]$. Viewing A as an A-module, find all of its 1-dimensional submodules.

48. Prove Claim 3.2.9 from the notes.

49. (a) *(Trivial representations don't generalize).* Let A be an algebra, and let M be a vector space. Suppose we try and define an A-module structure on M by the rule

$$ax = x, \qquad \forall a \in A,\, x \in M$$

What is wrong with this definition?

(b) *(Tensor product representations don't generalize).* Let M and N be modules, and let $\{x_1, ..., x_m\}$ and $\{y_1, ..., y_n\}$ be bases for M and N (as vector spaces). Suppose we try and define an A-module structure on the vector space $M \otimes N$ by the rule

$$a(x_i \otimes y_j) = a(x_i) \otimes a(y_j)$$

for all $a \in A$. What is wrong with this definition?

50. (a) Let $f : A \to B$ be an algebra homomorphism. Show that the same linear map defines an algebra homomorphism $f : A^{op} \to B^{op}$.

(b) Let M be an A-module. Show that $M^* = \mathrm{Hom}(M, \mathbb{C})$ is naturally an A^{op}-module.

51. Show that there are no homomorphisms from $\mathrm{Mat}_{2\times 2}(\mathbb{C})$ to $\mathrm{Mat}_{3\times 3}(\mathbb{C})$.

52. Let $G = C_4$. Write down an isomorphism from $\mathbb{C}[G]$ to a direct sum of matrix algebras. Do the same for $G = D_4$.

53. Prove that every commutative simple-simple algebra is isomorphic to $\mathbb{C}[G]$ for some group G.

54. Find an example of a pair of groups G and H such G and H are not isomorphic, but $\mathbb{C}[G]$ and $\mathbb{C}[H]$ are isomorphic algebras. Now find infinitely-many such pairs.

55. Let A be the 3-dimensional algebra $\langle 1, x, x^2 \rangle$ where $x^3 = 0$.

(i) Write down the homomorphism

$$A \to \mathrm{Mat}_{3\times 3}(\mathbb{C})$$

that we get by viewing A as an A-module, and using the given basis.

(ii) Show that A is not a semi-simple algebra.

56. Let A be the subspace

$$A = \left\{ \begin{pmatrix} x & y \\ 0 & z \end{pmatrix},\ x, y, x \in \mathbb{C} \right\} \subset \mathrm{Mat}_{2\times 2}(\mathbb{C})$$

(a) Verify that A is a subalgebra.

(b) Find a 2-dimensional A-module which is not semi-simple.

(c) Find a 2-dimensional subalgebra $B \subset A$ which is semi-simple.

57. Let $G = S_3$, and let U_1 and U_3 be the trivial and 2-dimensional irreps of G. Let A be the algebra

$$A = \mathrm{Hom}_G(U_1 \oplus U_3,\, U_1 \oplus U_3)$$

Find a subalgebra of $\mathrm{Mat}_{3\times 3}$ that is isomorphic to A. Is A semi-simple?

58. Up to isomorphism, how many semi-simple algebras are there which have dimension 7?